

Inelastic nucleon-deuteron scattering. II. Physical discussion of the impulse approximation with the double scattering terms included

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With 3 figures in the text

ABSTRACT

The treatment of inelastic nucleon-deuteron scattering is made more symmetric with respect to the particles in the final state by redefining the transition operator and including the double scattering terms in the impulse approximation. An estimate of the importance of the double scattering terms is made, and it is found that double scattering may be important at all energies, depending on the magnitude of final, rather than initial, relative momenta.

I. Introduction

Part I [1] of this work was mainly concerned with the derivation of formulae for the transition matrix elements in the conventional impulse approximation [2-5], for general two-body interactions. In connection with this the implications of the Pauli principle and the Coulomb interaction were also considered. Here we wish to discuss the inelastic nucleon-deuteron scattering from a physical point of view. For this purpose we shall first analyze the usual definition of the final state and reformulate the treatment in such a manner that all physical processes that may occur in the scattering are included in the transition operator, and the final state can be defined as that of three free nucleons.

Then we shall examine the double scattering corrections to the impulse approximation. These corrections are of great interest for many reasons. There is first the obvious question of their magnitude, which has been estimated earlier by several authors [3, 6, 7] who, however, arrive at rather different conclusions as to the importance of these terms. Therefore we have here made a simple independent estimate of these terms.

Another reason for the inclusion of double scattering terms in the impulse approximation for the nucleon-deuteron transition matrix is that Chew, in his presentation [2] of the impulse approximation, gives different alternatives for the approximation formulae, and to determine the proper alternative to be used one has to compare the relative momenta of the three pairs of nucleons in the final state. If one includes the double scattering terms, all these alternatives are automatically taken into account in a natural way.

A third reason for studying the double scattering terms is that inclusion of them will make it possible to describe the scattering in a unified manner, especially in combination with the revised definition of the final state mentioned above. This is of great value in visualizing the physical processes involved.

The errors due to the neglect of the binding forces in the deuteron have been discussed in detail by Chew and Wick [3], who estimate them to be only a few percent at energies above 100 MeV. They will accordingly be neglected here, and so will the multiple scattering terms of third or higher order, for reasons given at the end of section III.

II. Some remarks on the definition of the final state in inelastic nucleon-deuteron scattering

In a strictly stationary treatment of nucleon-deuteron scattering one has to define the final state of the deuteron in the inelastic case as a two-body continuum state in order to ensure that the final state does not contain any components of the deuteron ground state. As was pointed out in part I, such a definition of the final state is perhaps not quite natural from an experimental point of view, since all measurements of scattered intensities, polarizations etc. are performed in the asymptotic region when so long a time has elapsed after the scattering event that the wave function has attained its asymptotic form. Therefore we define, as we indicated in part I, eq. (34), an operator

$$T'_{1d} = \Omega_{23}^{(-)\dagger} T_{1d} \quad (1)$$

Here T_{1d} is the usual transition operator, which describes the scattering of the incoming particle 1 by the deuteron d , and $\Omega_{23}^{(-)}$ is the Møller wave operator, relating the final plane wave state $|\mathbf{k}'_2, \mathbf{k}'_3\rangle$ of the particles 2 and 3 to the state $|\Psi_{\mathbf{k}'_2, \mathbf{k}'_3}^{(-)}\rangle$ immediately after the scattering according to the equation

$$|\Psi_{\mathbf{k}'_2, \mathbf{k}'_3}^{(-)}\rangle = \Omega_{23}^{(-)} |\mathbf{k}'_2, \mathbf{k}'_3\rangle \quad (2)$$

It is clear that using the operator T'_{1d} with a plane wave final state is exactly equivalent to using the operator T_{1d} with the corresponding continuum final state. Remembering that the hermitean adjoint of the wave operator $\Omega_{23}^{(-)}$ describes the time evolution of the system [8], due to the interaction between the particles 2 and 3, from the time $t = 0$ of the scattering to the time $t = \infty$ of the physical measurements, we see that the form (1) of the transition operator is rather satisfactory from the experimental point of view. The operator T'_{1d} contains, actually, all dynamic processes in the (inelastic) scattering up to the measurement. We will also find this operator very useful in the following discussion of the impulse approximation, especially if the double scattering corrections are to be included.

III. Double scattering corrections to the impulse approximation

The formal derivation of the impulse approximation was first made in a simple case by Chew and Wick [3] and also by Ashkin and Wick [4]. Somewhat later Chew and Goldberger [5] published a very general derivation of the approximation, valid

in principle for scattering by a system with an arbitrary number of particles. In their derivation the exact transition operator is split into three parts, one of which is equal to the sum of all the two-body transition operators for the incident particle and one particle in the target system. This part corresponds to the usual impulse approximation. Of the other two parts, one is linear in the binding potential of the target system whereas the remaining part contains terms independent of the binding potential. In the limit of vanishing binding potential these terms would still be present in the transition operator, and it is natural to interpret them as multiple scattering terms. We will now examine these terms for the nucleon-deuteron case.

In our notation the expression (28) of reference [5] reads, including only double scattering terms,

$${}^d T_{1d} = T_{12}^{(-)\dagger} (\Omega_{13}^{(+)} - 1) + T_{13}^{(-)\dagger} (\Omega_{12}^{(+)} - 1) \quad (3)$$

where

$$T_{1j}^{(-)} = V_{1j} \Omega_{1j}^{(-)} \quad (4)$$

V_{1j} is the potential for the particles 1 and j . The factors

$$\Omega_{1k}^{(+)} - 1 \quad (5)$$

represent outgoing waves from a collision between the particles 1 and k , and it is natural to regard the operators $T_{1j}^{(-)\dagger}$ as representing a subsequent collision between the particles 1 and j .

Another type of processes involving two collisions, which one intuitively expects to occur about as often as the double scattering processes represented by (3), is a process in which particle 1 collides with 2, which then collides with 3. However, this process has already been accounted for, since by defining the final state of the deuteron as a continuum state we have allowed for the interaction between the particles 2 and 3 after the first collision. Actually, rewriting eq. (1) as

$$T'_{1d} = T_{1d} + (\Omega_{23}^{(-)\dagger} - 1) T_{1d} \quad (6)$$

we find in the second term of the right member the factor

$$\Omega_{23}^{(-)\dagger} - 1 \quad (7)$$

which is similar in form to the factors (5). We note that this factor represents the whole interaction between the particles 2 and 3 after the nucleon-deuteron collision, and we can describe this interaction as a "collision". This interpretation is justified by the fact that the operator (7) is proportional to the transition operator $T_{23}^{(-)\dagger}$, which can be seen, for instance, by representing $\Omega_{23}^{(-)}$ as

$$\Omega_{23}^{(-)} = \lim_{\varepsilon \rightarrow +0} \left\{ 1 + \frac{1}{E - i\varepsilon - K} T_{23}^{(-)} \right\} \quad (8)$$

which is valid when $\Omega_{23}^{(-)}$ operates on a plane wave state for the particles 2 and 3 with (positive) energy eigenvalue E . K is the kinetic energy operator for the two particles. Eq. (8) turns into the usual integral equation (see, e.g., reference [5]) for $\Omega_{23}^{(-)}$ if one replaces $T_{23}^{(-)}$ by $V_{23} \Omega_{23}^{(-)}$ according to (4).

Now we introduce the impulse approximation with inclusion of the double scattering terms (3) by writing

$$T_{1d} = T_{12} + T_{13} + {}^d T_{1d} \quad (9)$$

Inserting this in (6) we get

$$T'_{1d} = T_{12} + T_{13} + (\Omega_{23}^{(-)\dagger} - 1)(T_{12} + T_{13}) + T_{12}^{(-)\dagger}(\Omega_{13}^{(+)} - 1) + \\ + T_{13}^{(-)\dagger}(\Omega_{12}^{(+)} - 1) + (\Omega_{23}^{(-)\dagger} - 1){}^d T_{1d} \quad (10)$$

The last term in (10), viz.

$$(\Omega_{23}^{(-)\dagger} - 1){}^d T_{1d} \quad (11)$$

is of the third order in the two-body transition operators, that is, of third order or higher in the nucleon-nucleon potentials. It can be interpreted as corresponding to triple collision processes in which particle 1 first collides with 2 and 3 and then 2 and 3 collide with each other. If we are to include only double collision processes in the impulse approximation, it is consistent to neglect the term (11) in the expansion (10). This is also consistent with our neglect of the binding force, which, too, gives contributions of third order or higher in the nucleon-nucleon potentials, as is easily seen from Chew's and Goldberger's expression (24), reference [5], for the binding force corrections.

IV. Evaluation of the matrix elements of the approximate transition operator

The matrix elements of the transition operator T'_{1d} in the approximation (10) are easily evaluated employing the methods described in part I. The initial state is, as before,

$$|i\rangle = |\mathbf{k}_1^i, \Phi_0, \chi^i\rangle \quad (12)$$

where $\mathbf{k}_1^i$ is the initial wave vector of particle 1, Φ_0 is the deuteron ground state in the laboratory system, and χ^i is the initial three-body spin state. The final state is, in our description, a plane wave state

$$|f\rangle = |\mathbf{k}_1^f, \mathbf{k}_2^f, \mathbf{k}_3^f, \chi^f\rangle \quad (13)$$

where $\mathbf{k}_j^f$ is the final wave vector of the particle j ($j = 1, 2, 3$), and χ^f is the final spin state. In the following we shall exclude the spin states from the notations, for the sake of brevity.

It is convenient to introduce the relative and total momenta, $\hbar \mathbf{k}_{ij}$ and $\hbar \mathbf{K}_{ij}$ respectively, for the particles i and j , defined by

$$\mathbf{k}_{ij} = (\mathbf{k}_i - \mathbf{k}_j)/2 \\ \mathbf{K}_{ij} = \mathbf{k}_i + \mathbf{k}_j \quad (14)$$

Since the two-body operators occurring in (10) depend only on the relative momenta we can write

$$(\mathbf{k}'_i, \mathbf{k}'_j | Q_{ij} | \mathbf{k}_i, \mathbf{k}_j) = \delta(\mathbf{K}'_{ij} - \mathbf{K}_{ij}) (\mathbf{k}'_{ij} | q_{ij} | \mathbf{k}_{ij}) \quad (15)$$

where Q_{ij} stands for any one of the operators T_{ij} , $T_{ij}^{(-)}$, or $\Omega_{ij}^{(\pm)}$, and q_{ij} denotes the corresponding operator in relative momentum space only: t_{ij} , $t_{ij}^{(-)}$, and $\omega_{ij}^{(\pm)}$ respectively. Eq. (15) is an expression for the conservation of total momentum.

The matrix element in momentum space of the terms in (10) between the states (12) and (13) can be evaluated as in part I. We may also write them down directly with reference to some simple diagrams (Figs. 1, 2, 3) using eq. (15) for the two-body operators and the representative in momentum space of the initial state:

$$(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 | i) = \delta(\mathbf{k}_1 - \mathbf{k}_1^i) \delta(\mathbf{k}_2 + \mathbf{k}_3) g_0 \left(\frac{\mathbf{k}_2 - \mathbf{k}_3}{2} \right) \quad (16)$$

Here $g_0(\mathbf{k}_{23})$ is the wave function of the deuteron ground state in relative momentum space. The deuteron is supposed to be initially at rest in the laboratory system. The diagrams illustrate the physical processes that correspond to three typical terms in (10).

Fig. 1 shows a pure single scattering process, in which the incident particle 1 collides with particle 2 of the deuteron. The whole momentum transfer is taken up by 2, which leaves particle 3 behind with a momentum $\hbar \mathbf{k}_3^f = \hbar \mathbf{k}_3$, the momentum of 3 before the collision. As the deuteron was at rest before the collision, we have also $\mathbf{k}_2 = -\mathbf{k}_3$. This momentum is present in the ground state with a probability amplitude $g_0(-\mathbf{k}_3)$, and we get for the contribution to the transition operator T'_{1d} from the term T_{12}

$$(f | T_{12} | i) = \delta(\mathbf{k}_1^f + \mathbf{k}_2^f + \mathbf{k}_3^f - \mathbf{k}_1^i) \left(\frac{\mathbf{k}_1^f - \mathbf{k}_2^f}{2} | t_{12} | \frac{\mathbf{k}_1^i + \mathbf{k}_3^f}{2} \right) g_0(-\mathbf{k}_3^f) \quad (17)$$

where the delta function is an expression for the conservation of the total momentum for the three particles.

Next we consider the term $(\Omega_{23}^{(-)\dagger} - 1) T_{12}$, illustrated in Fig. 2. The incident particle 1 collides with particle 2 of the deuteron and leaves the scattering region. Then particle

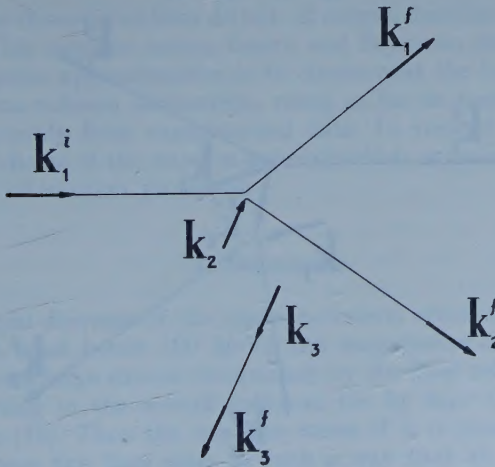


Fig. 1. Single scattering corresponding to the term T_{12} in equation (10).

2 collides with particle 3, sharing with 3 the momentum transfer taken up in the first collision. Thus we have an intermediate state $\mathbf{k}'_2$, which we have to integrate over to get the contributions from all processes compatible with the final state considered. Using momentum conservation we get for the first collision

$$\mathbf{k}_2 = \mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1^i$$

and for the second collision

$$\mathbf{k}_3 = \mathbf{k}'_2 + \mathbf{k}'_3 - \mathbf{k}'_2$$

As in the preceding case we have $\mathbf{k}_2 = -\mathbf{k}_3$. The transition matrix element is, then, for this process,

$$\begin{aligned} \langle f | (\Omega_{23}^{(-)\dagger} - 1) T_{12} | i \rangle = \\ = \delta(\mathbf{k}'_1 + \mathbf{k}'_2 + \mathbf{k}'_3 - \mathbf{k}_1^i) \int d^3 k'_2 \left(\frac{\mathbf{k}'_2 - \mathbf{k}'_3}{2} \middle| \omega_{23}^{(-)\dagger} - 1 \middle| \frac{\mathbf{k}'_2 + \mathbf{k}'_2 - \mathbf{k}'_2 - \mathbf{k}'_3}{2} \right) \times \quad (18) \\ \times \left(\frac{\mathbf{k}'_1 - \mathbf{k}'_2}{2} \middle| t_{12} \middle| \frac{\mathbf{k}_1^i - \mathbf{k}'_2 + \mathbf{k}'_2 + \mathbf{k}'_3}{2} \right) g_0(\mathbf{k}'_2 - \mathbf{k}'_2 - \mathbf{k}'_3) \end{aligned}$$

By setting $\mathbf{k}'_2 = \mathbf{k}'_3 - \mathbf{k}'_3/2$ we can put eq. (18) in the same form as eq. (18) of part I. The results are seen to be equivalent in virtue of eq. (2), remembering that here we have separated the contribution (17).

Finally we consider the term $T_{13}^{(-)\dagger} (\Omega_{12}^{(+)} - 1)$ in (10), illustrated in Fig. 3. The incident particle 1 collides first with particle 2, then with particle 3 of the deuteron. We have an intermediate state characterized by $\mathbf{k}'_1$ to integrate over. Momentum conservation gives

$$\mathbf{k}_2 = \mathbf{k}'_1 + \mathbf{k}'_2 - \mathbf{k}_1^i$$

and

$$\mathbf{k}_3 = \mathbf{k}'_1 + \mathbf{k}'_3 - \mathbf{k}'_1$$

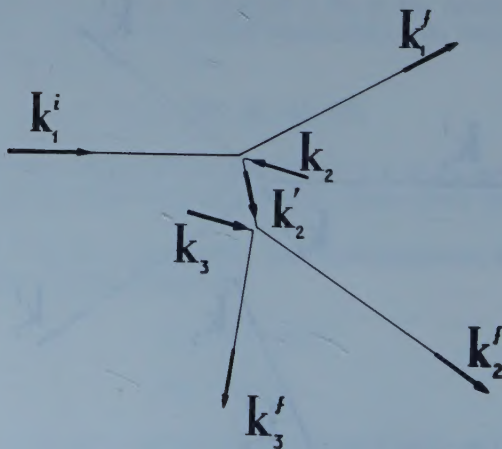


Fig. 2. Double scattering corresponding to the term $(\Omega_{23}^{(-)\dagger} - 1) T_{12}$ in equation (10).

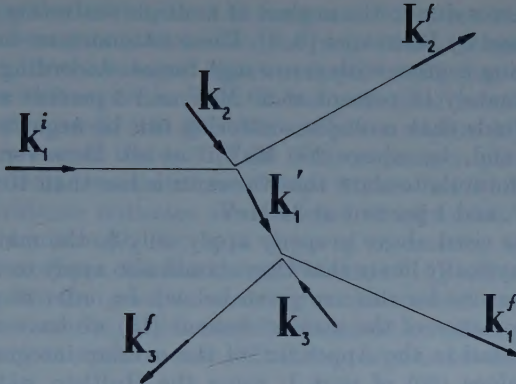


Fig. 3. Double scattering corresponding to the term $T_{13}^{(-)\dagger} (\Omega_{12}^{(+)} - 1)$ in equation (10).

for the first and the second collision, respectively. As before we have $\mathbf{k}_2 = -\mathbf{k}_3$. The matrix element for this process is

$$\begin{aligned}
 \langle f | T_{13}^{(-)\dagger} (\Omega_{12}^{(+)} - 1) | i \rangle = & \\
 = \delta(\mathbf{k}_1' + \mathbf{k}_2' + \mathbf{k}_3' - \mathbf{k}_1^i) \int d^3 k_1' \left(\frac{\mathbf{k}_1' - \mathbf{k}_3'}{2} \middle| t_{13}^{(-)\dagger} \middle| \frac{\mathbf{k}_1' - \mathbf{k}_1^i - \mathbf{k}_3' + \mathbf{k}_1'}{2} \right) \times & \quad (19) \\
 \times \left(\frac{\mathbf{k}_1' - \mathbf{k}_2'}{2} \middle| \omega_{12}^{(+)} - 1 \middle| \frac{\mathbf{k}_1^i + \mathbf{k}_1' + \mathbf{k}_3' - \mathbf{k}_1'}{2} \right) g_0(\mathbf{k}_1' - \mathbf{k}_1^i - \mathbf{k}_3')
 \end{aligned}$$

The matrix elements of the other terms in (10) (excepting the term (11) which we neglect in accordance with the remarks at the end of section III) may be evaluated in a similar manner or derived from (17), (18), and (19) by a suitable interchange of indices.

Diagrams such as those given here do not, of course, describe the scattering process in minute detail. This requires meson theory and Feynman diagrams. But the main purpose of the impulse approximation is to circumvent the lack of an exact knowledge of the nucleon-nucleon interaction, using as far as possible information that can be obtained directly from experimental data. In view of this our diagrams fit rather well in the scheme of the impulse approximation, representing the net effect of more complicated interaction processes.

V. Discussion

Owing to the rapid decrease of the nucleon-nucleon cross sections with increasing relative energy (at least below 100 MeV), the magnitudes of the matrix elements (18) and (19) are to a large extent determined by the final relative momenta of the particles participating in the second collision, i.e. by $\mathbf{k}_{23}' = (\mathbf{k}_2' - \mathbf{k}_3')/2$ in (18) and $\mathbf{k}_{13}' = (\mathbf{k}_1' - \mathbf{k}_3')/2$ in (19). Then the question arises if it is possible, e.g. in a (p, 2p) experiment, to choose the final state in such a way that all the double scattering terms can be neglected, only terms of the type (17) remaining.

Estimates of the error due to the neglect of multiple scattering have been made by Chew and Wick [3] and by Brueckner [6, 7]. These estimates are based on the assumption of fixed scattering centers with zero range forces. According to Chew and Wick the error is approximately 16 percent at 50 MeV and 3 percent at 200 MeV incident energy. Brueckner finds that multiple scattering can be neglected only if the Born approximation is valid, i.e. above 200 MeV if at all. However, Mather and Swan [7] use Brueckner's formula to show that the error is less than 10 percent at 25 MeV, 5 percent at 30 MeV, and 1 percent at 34 MeV.

Though the results cited above properly apply only to the matrix elements of the type (19) it seems physically likely that they should also apply to the matrix elements (18) (some formal reasons for this are given below). In order to get an independent estimate of the importance of the matrix element (18) we have made a calculation, described in more detail in the Appendix, of the overlap integral I_{23} , given by the second of the equations (20) of part I, using the Hulthén wave function for the ground state of the deuteron, and a plane wave plus s-wave for the continuum wave function. The contribution from the plane wave part is proportional to the matrix element (17), and the contribution from the s-wave part should be approximately proportional to (18). The approximations involved are certainly too crude for a serious calculation of (18), but for the purpose of estimating the error this method should be sufficient. In a sense it is actually more realistic than the methods used by Chew and Wick and by Brueckner, as the scattering centers are no longer supposed to be fixed.

The results obtained here may be summarized as follows: For large values of $|\mathbf{k}'_{23}|$ the matrix element (18) is small, but if simultaneously $|\mathbf{k}'_3|$ is large, (17) is also small, so that in this case the relative error can be large. For small values of $|\mathbf{k}'_3|$ our results are similar to those cited above for the "genuine" double scattering terms, depending, however, on final, rather than initial, relative momenta.

In this connection we wish to discuss briefly another question. Chew [2] gives the following recipe for the practical use of the impulse approximation: Define as final state $|f\rangle$ in the term $\langle f|T_{1k}|i\rangle$ in the impulse approximation to $\langle f|T_{1a}|i\rangle$ a product of a continuum state for that pair of particles (different from the pair (1, k)) which has the smallest final relative momentum. In our formalism this recipe can be written (cf. eq. (7))

$$T'_{1a} = T_{12} + T_{13} + \frac{(\Omega_{23}^{(-)\dagger} - 1)}{(\Omega_{13}^{(-)\dagger} - 1)} T_{12} + \frac{(\Omega_{23}^{(-)\dagger} - 1)}{(\Omega_{12}^{(-)\dagger} - 1)} T_{13} \quad (20)$$

where the upper or the lower alternative is to be chosen according to the magnitudes of the final relative momenta. Now the difference between $(\Omega_{13}^{(-)\dagger} - 1)T_{12}$ in (20) and $T_{13}(\Omega_{12}^{(-)\dagger} - 1)$ in (10) is less than the remaining error in the impulse approximation (see refs. [4] or [5]). The Chew recipe is thus nearly equivalent to neglecting the two smaller double scattering terms in (10). On the other hand, retaining all the alternatives in (20) one would get an approximation to (10). In view of these remarks the results obtained in the appendix for the term (18) should apply at least approximately to the term (19), since both upper and lower alternative in (20) is of the same form, leading to matrix elements of the type (18).

We conclude this section with some comments on another approximation used by Chew [2] and also in part I of this work, namely that of factorizing the matrix elements (18) and (19) by moving the t -matrix element outside the integral. Chew's

argument is based on the fulfilment of at least one of two conditions, namely (i) that the t -matrix elements vary only little over the range of the integration variable where the other factors under the integral sign are large, or (ii) that the matrix elements depend only on the difference between initial and final momenta. The condition (ii) is useful only in the case of (18). For (19) one has to require dependence on the *sum* of initial and final momenta instead. If the nucleon-nucleon potentials are non-local (e.g. velocity-dependent) the conditions of type (ii) are hardly at hand, and actually the experimental evidence indicates the presence of spin-orbit coupling and also a hard core in the potentials [9, 10], both types of interaction separately leading to non-locality. (Cf. Brueckner, Gammel, and Weitzner [11]; although their investigation is concerned with the reaction matrix for the nucleon-nucleon interaction in nuclear matter, these results, which may be surmised on other grounds, should also apply in the free nucleon-nucleon case.) Therefore arguments based on condition (ii) are not very reliable.

With respect to condition (i) the situation is more obscure. In part I we used the fact that the continuum function $g_{k_{23}}^f(k_{23})$ contains a delta function singularity. In (18) this delta function is subtracted off. The remainder is proportional to the matrix element $(k_{23}|t_{23}^{-1}|k_{23})$, as is easily seen using the representation (8). If the t -matrix elements were sharply peaked in the forward direction ($k_{23} = k_{23}^f$) the argument could still be used, but this seems not to be the case, judging for instance from the t -matrix elements calculated by Squires [12] from the Gammel-Thaler potentials [9]. However, if the matrix element (17), the factorization of which is, of course, exact and in the desired manner, gives the main contribution to the nucleon deuteron matrix elements, and the double scattering terms are small, then it may still be considered a fairly good approximation to factorize the matrix elements (18) and (19) in order to simplify calculations.

VI. Conclusions

In this study of the inelastic nucleon-deuteron scattering we have found that at high incident energies, where the binding forces in the deuteron may be neglected, the impulse approximation is a simple, useful and fairly accurate tool, provided the two-body wave functions for bound and scattering states are known. Multiple scattering may be important at all energies, since it depends on the magnitude of final, rather than initial, relative momenta. The inclusion of the double scattering terms presents no formal difficulties but requires knowledge of the two-body continuum wave functions. In the treatment of the various spin states the methods described in part I are readily applicable.

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APPENDIX

The overlap integral I_{23} , defined by (cf. eq. (20) of part I)

$$I_{23}(\mathbf{k}, \mathbf{K}) = \int d^3 k' g_{\mathbf{k}}^*(\mathbf{k}') g_0(\mathbf{k}' - \mathbf{K}) \quad (\text{A1})$$

has been calculated with the Hulthén wave function [13, 14]

$$\varphi_0(r) = N(e^{-\alpha r} - e^{-\beta r})/r \quad (\text{A2})$$

for the deuteron ground state, and a plane wave plus s -wave

$$\psi_{\mathbf{k}}^{(-)}(\mathbf{r}) = C(e^{+i\mathbf{k}\cdot\mathbf{r}} + A(k)e^{-ikr}/r) \quad (\text{A3})$$

for the neutron-proton continuum state with final relative momentum $\mathbf{k}$. N and C are normalization constants, α and β are parameters, depending on the binding energy of the deuteron and the range of the neutron-proton interaction (see, e.g., ref. [13]). The scattering amplitude is chosen according to the shape-independent approximation [15, 7] as

$$A(k) = 1/(-1/a - ik + r_0 k^2/2) \quad (\text{A4})$$

which follows from the relations $A(k) = (e^{2i\delta_0} - 1)/2ik$ and $k \cot \delta_0 = -1/a + r_0 k^2/2$, the usual expression for the shape-independent approximation. a is the scattering length, r_0 the effective range.

The overlap integral I_{23} can be computed directly from the position space wave functions according to (cf. Chew [2] eq. (20))

$$I_{23} = \int d^3 r e^{i\mathbf{K}\cdot\mathbf{r}} \varphi_0(\mathbf{r}) \psi_{\mathbf{k}}^{(-)*}(\mathbf{r}) \quad (\text{A5})$$

From (A2), (A3), and (A5) we get

$$I_{23}/NC = 4\pi g(\mathbf{k} - \mathbf{K}) + 2\pi A^*(k) h(k, K)/iK \quad (\text{A6})$$

where

$$g(\mathbf{p}) = 1/(\alpha^2 + p^2) - 1/(\beta^2 + p^2) \quad (\text{A7})$$

which is proportional to $g_0(\mathbf{p})$, and

$$\begin{aligned} h(k, K) = \frac{1}{2} \ln \frac{(\alpha^2 + (k+K)^2)(\beta^2 + (k-K)^2)}{(\alpha^2 + (k-K)^2)(\beta^2 + (k+K)^2)} + \\ + i \left(\arctg \frac{2\beta K}{\beta^2 - K^2 + k^2} - \arctg \frac{2\alpha K}{\alpha^2 - K^2 + k^2} \right) \end{aligned} \quad (\text{A8})$$

For $\mathbf{k} = \mathbf{k}_{23}^f$ and $\mathbf{K} = \mathbf{K}_{23}^f/2$, eqs. (14) yield

$$\mathbf{k} - \mathbf{K} = -\mathbf{k}_3^f$$

Evidently the first term in the right member of (A6) is proportional to the matrix element (17), and the second term is approximately proportional to (18). As a measure of the relative error in neglecting (18) we have chosen the quantity

$$Q = |A(k) h(k, K) / 2Kg(\mathbf{k} - \mathbf{K})| \quad (\text{A9})$$

The deuteron parameters were taken from Moravsik [14]:

$$\alpha = 0.232 \times 10^{13} \text{ cm}^{-1}$$

$$\beta = 1.202 \times 10^{13} \text{ cm}^{-1}$$

The shape-independent parameters were taken from ref. [7], p. 204 et seqq.:

$$a = 0.522 \times 10^{-12} \text{ cm}$$

$$r_0 = 1.6 \times 10^{-13} \text{ cm}$$

The quantity Q was calculated with these parameters for some special choices of K , k , and $k'_3 = |\mathbf{k} - \mathbf{K}|$, which were expected to give relatively large Q values. The results are tabulated below:

Table 1. The quantity Q according to eqs. (A6)–(A9).

k'_3 10^{13} cm^{-1}	$K=0$			$K=k$		
	$k=2.5 \alpha$	$k=5 \alpha$	$k=7.5 \alpha$	$k=2.5 \alpha$	$k=5 \alpha$	$k=7.5 \alpha$
0.0	0.198	0.0183	0.0055	0.109	0.0257	0.0092
0.2	0.354	0.0327	0.0098	0.196	0.0458	0.0165
0.4	0.871	0.0805	0.0242	0.483	0.113	0.0407
0.6	1.89	0.175	0.0526	1.05	0.245	0.0883
0.8	3.78	0.341	0.102	2.04	0.476	0.172
1.0	5.57	0.514	0.155	3.09	0.723	0.270

The values of the final relative momentum $k=2.5\alpha$, 5α , and 7.5α correspond to a final relative energy of approximately 14, 55, and 122 MeV respectively. The shape-independent approximation is, of course, not accurate for the higher energies, where also higher order partial waves contribute.

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